Topological applications of long ω_1-approximation sequences

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2015 Winter School in Abstract Analysis
Hejnice, Czech Republic

A long time ago, some authors used "curve" to denote an isometric copy of a graph of a function $\mathbb{R} \to \mathbb{R}$. (Continuity is not required.)

If such a curve is a measurable subset of $\mathbb{R}^2$, then it is null.

However, Sierpiński showed (1933) that, assuming CH, the plane is a countable union of graphs of functions and their converses:

- Let $\lhd$ order $\mathbb{R}$ with type ω_1.
- Let f_x map ω onto $\{y : y \unlhd x\}$.
- Let $g_n(x) = f_x(n)$.
- $\bigcup_{n<\omega}(g_n \cup g_n^{-1}) = \bigcup_{n<\omega}\bigcup_{x\in\mathbb{R}}\{(x, g_n(x)), (g_n(x), x)\} = \mathbb{R}^2$

Thus, CH implies that the plane is a countable union of curves.

Sierpiński asked (1951) if CH is needed to cover the plane by countably many curves.

Roy O. Davies answered "no" (1963) with an ingenious ZFC covering. (Never underestimate the axiom of choice!)

To cover the plane by countable many curves, it is enough to partition the plane into countably many partial curves.

Fix an ω-sequence pairwise non-parallel lines $(L_n : n < \omega)$. (For us, identical lines are considered parallel.)

Davies constructed a partition $\bigsqcup_{n<\omega} C_n = \mathbb{R}^2$ such that $|L \cap C_n| \leq 1$ for all n and all lines $L||L_n$.

(Davies remarked that an argument of Sierpiński implicitly shows that, given a covering of $\mathbb{R}^2$ by countably many curves, there is a covering of $\mathbb{R}^2$ by countably many pairwise isometric curves.)

To a set theorist, the tastiest ingredient of Davies' proof is his following implicit lemma.

Lemma (Davies' Lemma). *Let $\mathcal{L}$ be a countable first order language. Let $\mathfrak{A}$ be an uncountable $\mathcal{L}$-structure.* **Then** *there is a transfinite sequence $\overline{\mathfrak{M}} = (\mathfrak{M}_\alpha)_{\alpha<\eta}$ such that*

- *every $\mathfrak{M}_\alpha$ is a countable substructure of $\mathfrak{A}$,*

- $\bigcup \mathrm{ran}(\overline{\mathfrak{M}}) = \mathfrak{A}$, *and*

- $\overline{\mathfrak{M}}$ *has the* **Davies property***: for all $\alpha \leq \eta$,*

  $\mathfrak{M}_{<\alpha} = \bigcup_{\beta<\alpha} \mathfrak{M}_\beta$ *is a* **finite** *union of substructures of $\mathfrak{A}$.*

Davies' partition of the plane applies his lemma to a partial Skolemization of $(\mathscr{P}, \mathscr{L}, \in; L_n : n < \omega)$ where $\mathscr{P}$ is the set $\mathbb{R}^2$ of points in the plane and $\mathscr{L}$ is the set of lines in the plane.

We will simply let $\mathfrak{A}$ be a complete Skolemization of $(\mathscr{P}, \mathscr{L}, \in; L_n : n < \omega)$. Therefore, all substructures are elementary substructures.

Let $\overline{\mathfrak{M}} = (\mathfrak{M}_\alpha)_{\alpha<\eta}$ be as in Davies' Lemma.

Suppose that $\alpha < \eta$ and we have constructed a partition $\bigsqcup_{n<\omega} C_n = \mathscr{P} \cap \mathfrak{M}_{<\alpha}$ such that $|L \cap C_n| \leq 1$ for all n and all lines $L || L_n$.

It suffices to show that we can extend $\overline{C}$ to a partition $\bigsqcup_{n<\omega} C_n''' = \mathscr{P} \cap \mathfrak{M}_{<\alpha+1}$ such that $|L \cap C_n'''| \leq 1$ for all n and all lines $L || L_n$.

Let $\nu \leq \omega$ and let $\overline{p} = (p_k)_{k<\nu}$ biject from ν to $\mathscr{P} \cap \mathfrak{M}_\alpha \setminus \mathfrak{M}_{<\alpha}$.

Suppose that $k < \nu$ and we have extended $\overline{C}$ to a partition $\bigsqcup_{n<\omega} C'_n = \mathscr{P} \cap \mathfrak{M}_{<\alpha} \cup \{p_j : j < k\}$ such that $|L \cap C'_n| \leq 1$ for all n and all lines $L || L_n$.

It suffices to show that that we can extend $\overline{C}'$ to a partition $\bigsqcup_{n<\omega} C''_n = \mathscr{P} \cap \mathfrak{M}_{<\alpha} \cup \{p_j : j < k+1\}$ such that $|L \cap C''_n| \leq 1$ for all n and all lines $L || L_n$.

Let $d < \omega$ and $\overline{\mathfrak{N}} = (\mathfrak{N}_i)_{i<d}$ be such that $\mathfrak{M}_{<\alpha} = \bigcup \mathrm{ran}(\overline{\mathfrak{N}})$ and each $\mathfrak{N}_i$ is a substructure of $\mathfrak{A}$.

For each $n < \omega$, let K_n be the line through p_k that is parallel to L_n.

It suffices to show that there exists $n < \omega$ such that K_n is disjoint from $\mathfrak{M}_{<\alpha} \cup \{p_j : j < k\}$.

For each $j < k$, there is at most one $n < \omega$ such that $p_j \in K_n$.

For each $i < d$, there is at most one $n < \omega$ such that K_n intersects $\mathscr{P} \cap \mathfrak{N}_i$. Why? If $m < n < \omega$, $x \in K_m \cap \mathfrak{N}_i$, and $y \in K_n \cap \mathfrak{N}_i$, then $K_m, K_n \in \mathfrak{N}_i$; then $p_k \in \mathfrak{N}_i$ because $K_m \cap K_n = \{p_k\}$. But $p \notin \mathfrak{N}_i$.

Thus, K_n is disjoint from $\mathfrak{M}_{<\alpha} \cup \{p_j : j < k\}$ for almost all n. $\square$

Davies' Lemma apparently was not used in print again until 2002 by Jackson and Mauldin, and then by Milovich starting in 2008.

Jackson and Mauldin constructed (in ZFC) a Steinhaus set, that is, a subset of $\mathbb{R}^2$ that intersects every isometric copy of $\mathbb{Z}^2$ at exactly one point.

Without Davies' Lemma, Jackson and Mauldin's proof would have needed CH.

We do not know if higher-dimensional analogs of Steinhaus sets exist.

References

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How did Davies prove his lemma? Recall:

Lemma (Davies' Lemma)**.** *Let $\mathcal{L}$ be a countable first order language. Let $\mathfrak{A}$ be an uncountable $\mathcal{L}$-structure.* **Then** *there is a transfinite sequence $\overline{\mathfrak{M}} = (\mathfrak{M}_\alpha)_{\alpha<\eta}$ such that*

- *every $\mathfrak{M}_\alpha$ is a countable substructures of $\mathfrak{A}$,*
- $\bigcup \mathrm{ran}(\overline{\mathfrak{M}}) = \mathfrak{A}$, *and*
- $\overline{\mathfrak{M}}$ *has the* **Davies property***: for all $\alpha \leq \eta$,*

$$\mathfrak{M}_{<\alpha} = \bigcup_{\beta<\alpha} \mathfrak{M}_\beta \text{ is a \textbf{finite} union of substructures of } \mathfrak{A}.$$

Proof: The Davies tree. Recursively construct as follows a sequence $(\mathfrak{B}_t : t \in T)$ with T a subtree of $\mathrm{O}rd^{<\omega}$.

- $\mathfrak{B}_{()} = \mathfrak{A}$.
- If $\mathfrak{B}_t$ is countable, declare t to be a leaf of T.
- If $|\mathfrak{B}_t| = \kappa > \aleph_0$, declare $t^\frown(\alpha) \in T$ for all $\alpha < \kappa$ and choose an increasing sequence $(\mathfrak{B}_{t^\frown(\alpha)})_{\alpha<\kappa}$ of substructures of $\mathfrak{B}_t$ with union $\mathfrak{B}_t$ such that $|\mathfrak{B}_{t^\frown(\alpha)}| < |\mathfrak{B}_t|$ for all α.

T is well-founded. Therefore, the set L of leaves of T is well ordered by its lexicographic order $<_{\mathrm{lex}}$.

Moreover, $\bigcup_{t\in L} \mathfrak{B}_t = \mathfrak{A}$.

Finally, if $t \in L$, then $\bigcup_{s<_{\mathrm{lex}} t} \mathfrak{B}_s = \bigcup_{i<\mathrm{dom}(t)} \bigcup_{\alpha<t_i} \mathfrak{B}_{(t\restriction i)^\frown(\alpha)}$. $\square$

Note that if $|\mathfrak{A}| = \aleph_n < \aleph_\omega$, then the Davies tree has height $n+1$. Therefore:

Lemma. *Let $\mathcal{L}$ be a countable first order language. Let $\mathfrak{A}$ be an uncountable $\mathcal{L}$-structure* **of size** $\aleph_n < \aleph_\omega$. **Then** *there is a transfinite sequence $\overline{\mathfrak{M}} = (\mathfrak{M}_\alpha)_{\alpha<\eta}$ such that*

- *every $\mathfrak{M}_\alpha$ is a countable substructure of $\mathfrak{A}$,*
- $\bigcup \mathrm{ran}(\overline{\mathfrak{M}}) = \mathfrak{A}$, *and*
- *for all $\alpha \leq \eta$, $\mathfrak{M}_{<\alpha}$ is a union* **at most n** *substructures of $\mathfrak{A}$.*

For each cardinal κ, let $H(\kappa)$ denote the set of all sets x with transitive closure $\bigcup_{n<\omega} \bigcup^n x$ of cardinality less than κ.

For each regular uncountable cardinal θ, $(H(\theta), \in)$ is a model of ZFC except possibly for the power set axiom.

We will always implicitly choose θ large enough to include all the sets and power sets we need for the problem at hand.

The notation $N \prec H(\theta)$ means that N is an elementary $\{\in\}$-substructure of $H(\theta)$.

A *long ω_1-approximation sequence* is a transfinite sequence $\overline{M} = (M_\alpha)_{\alpha<\eta}$ of countable elementary substructures of $(H(\theta), \in)$ that is **retrospective**:

for each $\alpha < \eta$, the sequence $(M_\beta)_{\beta<\alpha}$ is an element of M_α.

Warning: If α is uncountable, then $(M_\beta)_{\beta<\alpha}$, $\{M_\beta : \beta < \alpha\}$, and $M_{<\alpha} = \bigcup_{\beta<\alpha} M_\beta$ are not subsets of M_α.

If $\overline{M}$ is a long ω_1-approximation sequence, $A \in M_0$, and $0 < \alpha < \mathrm{dom}(\overline{M})$, then M_0 and α are definable from $(M_\beta)_{\beta<\alpha}$, and hence elements of M_α.

Recall that if $X \in N \prec H(\theta)$ and $|X| \leq \aleph_0$, then $X \subset N$.

Therefore, $M_0 \subset M_\alpha$ for all $\alpha \in \mathrm{dom}(\overline{M})$.

Also, $M_\beta \subset M_\alpha$ for all $\beta \leq \alpha \in \omega_1 \cap \mathrm{dom}(\overline{M})$.

More generally, for all $\alpha, \beta \in \mathrm{dom}(\overline{M})$, we have

$$M_\beta \subsetneq M_\alpha \Leftrightarrow M_\beta \in M_\alpha \Leftrightarrow \beta \in \alpha \cap M_\alpha.$$

Recall that if $\mathfrak{A}$ is a first order structure for a countable language $\mathfrak{L}$ and $\mathfrak{A} \in N \prec H(\theta)$, then $\mathfrak{A} \cap N \prec_{\mathfrak{L}} \mathfrak{A}$.

Therefore, assuming $\mathfrak{A} \in M_0$, we have $\mathfrak{A} \cap M_\alpha \prec_{\mathfrak{L}} \mathfrak{A}$ for all $\alpha \in \mathrm{dom}(\overline{M})$.

Moreover, if every $M_{<\alpha}$ is a finite union of elementary substructures of $H(\theta)$ (and we will show that it is), then every $\mathfrak{A} \cap M_{<\alpha}$ is a finite union of $\mathfrak{L}$-elementary substructures of $\mathfrak{A}$.

Choose a surjection $f \colon |\mathfrak{A}| \to \mathfrak{A}$ in M_0. Assuming $|\mathfrak{A}| \leq \mathrm{dom}(\overline{M})$, we have $f(\alpha) \in M_\alpha$ for all $\alpha < |\mathfrak{A}|$. Therefore, $\bigcup_{\alpha<|\mathfrak{A}|}(\mathfrak{A} \cap M_\alpha) = \mathfrak{A}$.

Long ω_1-approximation sequences are canonical sequences of countable structures that are sufficiently rich to encode Davies trees of which they are leaves.

A Davies tree is built top-down, starting from a large structure. Long ω_1-approximation sequences are more flexibly built up from countable structures, which simplifies the construction of large structures "from scratch."

Long ω_1-approximation sequences provide a uniformly definable version of the Davies property and additional coherence properties.

The *cardinal normal form* of an ordinal α is the polynomial

$$\omega_{\beta_0} \cdot \gamma_0 + \omega_{\beta_1} \cdot \gamma_1 + \cdots + \omega_{\beta_{m-1}} \cdot \gamma_{m-1} + \gamma_m$$

that equals α and satisfies

- $\beta_0 > \cdots > \beta_{m-1} \geq 1$,
- $1 \leq \gamma_i < \omega_{\beta_i}^{+}$ for all $i < m$, and
- $\gamma_m < \omega_1$.

An example cardinal normal form:

$$\omega_{\omega+1} \cdot 4 + \omega_\omega + \omega_7 \cdot \left(\omega_7^{\omega_3^{\omega_7}} + \omega_6 \cdot \omega\right) + \omega_1 \cdot \omega_1 + (\omega^\omega + \omega \cdot 2 + 3)$$

The mapping sending each ordinal α to the code $\left(\overline{\beta}, \overline{\gamma}\right)$ for its unique cardinal normal form is uniformly definable without parameters according to the following computation.

- For every $\zeta \geq \omega_1$, let $\lfloor\zeta\rfloor$ be the greatest $|\zeta| \cdot \delta \leq \zeta$.
- For every $\zeta < \omega_1$, let $\lfloor\zeta\rfloor = \zeta$.
- For every ordinal ζ, let $\partial\zeta$ be the unique ε such that $\lfloor\zeta\rfloor + \varepsilon = \zeta$.
- For every ordinal ζ, let $\alpha_0 = \alpha$ and $\alpha_{i+1} = \partial\alpha_i$ for each $i < \omega$.
- For each $i < \omega$, let $\partial_i\alpha = \lfloor\alpha_i\rfloor$.
- Let m be least such that $\alpha_m < \omega_1$.
- For each $i < m$, let β_i satisfy $\omega_{\beta_i} = |\partial_i\alpha|$.
- For each $i < m$, let γ_i satisfy $\omega_{\beta_i} \cdot \gamma_i = \partial_i\alpha$.
- Let $\gamma_m = \partial_m\alpha$.

Given a cardinal normal form $\alpha = \sum_{i<m} \omega_{\beta_i} \cdot \gamma_i + \gamma_m$:

We have $\partial_i \alpha = \omega_{\beta_i} \cdot \gamma_i$ for each $i < m$ and $\partial_m \alpha = \gamma_m$.

Let $\lfloor \alpha \rfloor_i = \sum_{j<i} \partial_j \alpha$ for each $i \leq m$.

Let $\daleth(\alpha) = m + 1$ if $\gamma_m > 0$ and $\daleth(\alpha) = m$ if $\gamma_m = 0$.

Let $I_i(\alpha) = [\lfloor \alpha \rfloor_i, \lfloor \alpha \rfloor_{i+1})$ for all $i < \daleth(\alpha)$.

Fundamental Lemma. *If $(M_\alpha)_{\alpha<\eta}$ is a long ω_1-approximation sequence and $i < \daleth(\eta)$, then $\{M_\alpha : \alpha \in I_i(\eta)\}$ is directed (with respect to $\subset$). Hence, $\bigcup\{M_\alpha : \alpha \in I_i(\eta)\} \prec H(\theta)$.*

The lemma applies to every initial segment of $\overline{M}$. Therefore, $\overline{M}$ has (the analog of) the Davies property.

Proof. Proceed by induction on η.

• If $\eta \leq \omega_1$, then $I_i(\eta) = \eta$ and $\{M_\alpha : \alpha < \eta\}$ is a chain.

• If $\daleth(\eta) \geq 2$, then $\{M_\alpha : \alpha \in I_i(\eta)\}$ is directed by our induction hypothesis.

Why? First, $I_i(\eta) = [\lfloor\eta\rfloor_i, \lfloor\eta\rfloor_i + \partial_i\eta)$ and $I_0(\partial_i\eta) = \partial_i\eta < \eta$.

Second, $\lfloor\alpha\rfloor_i = \lfloor\eta\rfloor_i$ for all $\alpha \in I_i(\eta)$, so each M_α can compute a decomposition $\alpha = \lfloor\eta\rfloor_i + \beta$ from the cardinal normal of α, so $\left(M_{\lfloor\eta\rfloor_i+\beta}\right)_{\beta<\partial_i\eta}$ is retrospective.

• If $\eta = \kappa\cdot\gamma$ where κ is a an uncountable cardinal, γ is a limit ordinal, and $\gamma < \kappa^+$, then $I_i(\eta) = \eta$ and $\{M_\alpha : \alpha < \eta\}$ is directed because by our induction hypothesis $\{M_\alpha : \alpha < \kappa \cdot \beta\}$ is directed for all $\beta < \gamma$.

• The only remaining case is that $\eta = \kappa \cdot (\beta + 1)$ where κ is a an uncountable cardinal and $1 \leq \beta < \kappa^+$.

$M_{\kappa\cdot\beta}$ can compute κ and β from $\kappa\cdot\beta$ and then compute η. Therefore, $M_{\kappa\cdot\beta}$ knows that $|\eta| = \kappa$. Choose a surjection $f\colon \kappa \to \eta$ in $M_{\kappa\cdot\beta}$.

For each $\alpha < \kappa$, $M_{\kappa\cdot\beta+\alpha}$ knows the cardinal normal form $\kappa \cdot \beta + \alpha$. Hence, $f \in M_{\kappa\cdot\beta} \subset M_{\kappa\cdot\beta+\alpha}$ and $\alpha \in M_{\kappa\cdot\beta+\alpha}$; hence, $M_{f(\alpha)} \subset M_{\kappa\cdot\beta+\alpha}$.

Thus, $\{M_\alpha : \kappa \cdot \beta \leq \alpha < \eta\}$ is cofinal in $\{M_\alpha : \alpha < \eta\}$.

$\{M_\alpha : \kappa \cdot \beta \leq \alpha < \eta\}$ is directed by our induction hypothesis applied to $(M_{\kappa\cdot\beta+\alpha})_{\alpha<\kappa}$. $\square$

The Fundamental Lemma implies that every $M_{<\alpha}$ is the union of $\daleth(\alpha)$-many elementary substructures of $H(\theta)$.

By definition, $|I_{\daleth(\alpha)-2}(\alpha)| \geq \aleph_1$ and

$$|\alpha| = |I_0(\alpha)| > |I_1(\alpha)| > \cdots > |I_{\daleth(\alpha)-1}(\alpha)|.$$

Hence, if $1 \leq n < \omega$ and $\alpha < \omega_n$, then $\daleth(\alpha) \leq n$.

Therefore, for all $n \in [1, \omega)$ and all $\alpha < \omega_n$, $M_{<\alpha}$ is the union at most n elementary substructures of $H(\theta)$.

$n = 1$ is the trivial case where $\alpha < \omega_1$ and $M_{<\alpha} \prec H(\theta)$ because $\{M_\beta : \beta < \alpha\}$ is a chain.

Given a long ω_1-approximation sequence $(M_\alpha)_{\alpha<\eta}$, let:

- $M_{<\alpha} = \bigcup\{M_\beta : \beta < \alpha\}$ for each $\alpha \leq \eta$;
- $N^i_\alpha = \bigcup\{M_\alpha : \alpha \in I_i(\eta)\}$ for each $\alpha \leq \eta$ and $i < \daleth(\alpha)$;
- $P^i_\alpha = N^i_\alpha \cap M_\alpha$ for each $\alpha < \eta$ and $i < \daleth(\alpha)$.

By the Fundamental Lemma, $M_{<\alpha} = \bigcup_{i<\daleth(\alpha)} N^i_\alpha$ and $N^i_\alpha \prec H(\theta)$.

Some easily proved coherence properties:

Starting from $\overline{M} \restriction \alpha$, M_α can compute α, then $I_i(\alpha)$, and then N^i_α. Hence, $N^i_\alpha \in M_\alpha$ and, for every $n < \omega$, M_α knows that $N^i_\alpha \prec_{\Sigma_n} H(\theta)$. Hence, $P^i_\alpha \prec M_\alpha$.

If $j < i < \daleth(\alpha)$, then $\lfloor\lfloor\alpha\rfloor_i\rfloor_j = \lfloor\alpha\rfloor_j$, so $N^j_\alpha \in M_{\lfloor\alpha\rfloor_i} \subset P^i_\alpha \subset N^i_\alpha$.

Additional coherence properties of $(M_\alpha)_{\alpha<\eta}$:

• Each $\{M_\alpha : \alpha \in I_i(\eta)\}$ is a $\vee$-semilattice (with respect to $\subset$).

• For every nonempty $I \subset \eta$, there exists $J \subset \min(I) + 1$ such that $\bigcup_{\beta \in J} M_\beta$ is a directed union equal to $\bigcap_{\alpha \in I} M_\alpha$.

• For every nonempty $s \subset \daleth(\eta)$,

$$\bigcap_{i \in s} \{M_\alpha : \alpha < \eta \text{ and } \exists \beta \in I_i(\eta) \;\; M_\alpha \subset M_\beta\}$$

is directed.

• If $D \subset \eta$ and $\{M_\alpha : \alpha \in D\}$ is directed (and nonempty), then there exists $i < \daleth(\eta)$ such that for every $\alpha \in D$ there exists $\beta \in I_i(\eta)$ such that $M_\alpha \subset M_\beta$.

Suppose $\mathfrak{A}$ is an uncountable first order structure for a countable language $\mathfrak{L}$, $(M_\alpha)_{|\mathfrak{A}|}$ is a long ω_1-approximation sequence, and $\mathfrak{A} \in M_0$. We can recover a Davies tree from $\overline{M}$ as follows.

Let S denote the set of all $\alpha \leq |\mathfrak{A}|$ whose cardinal normal forms $\sum_{i<m} \omega_{\beta_i} \cdot \gamma_i + \gamma_m$ are such that $\gamma_{\daleth(\alpha)}$ is a successor ordinal.

Let $\mathcal{C}_\alpha = \mathfrak{A} \cap N_\alpha^{\daleth(\alpha)-1}$ for all $\alpha \in S$. (So $\mathcal{C}_{\beta+1} = M_\beta$ for all $\beta < |\mathfrak{A}|$.)

For each $\alpha \in S \cap |\mathfrak{A}|$, let

$$\alpha' = \begin{cases} \lfloor \alpha \rfloor_{\daleth(\alpha)-1} + |\partial_{\daleth(\alpha)-2}\alpha| & : \daleth(\alpha) \geq 2; \\ |\mathfrak{A}| & : \daleth(\alpha) = 1. \end{cases}$$

Let $\mathcal{T} = \{\mathcal{C}_\alpha : \alpha \in S\}$ and order $\mathcal{T}$ by declaring $\mathcal{C}_{\alpha'}$ to be the parent of $\mathcal{C}_\alpha$ for all $\alpha \in S \cap |\mathfrak{A}|$.

$\mathcal{T}$ is a tree with root $\mathfrak{A}$; nodes are leaves iff they are countable; the children of each non-leaf node $\mathcal{C}_\alpha$ are well-ordered by $\subset$, have cardinality less than $|\mathcal{C}_\alpha|$, and have union $\mathcal{C}_\alpha$.

Given a regular uncountable cardinal λ, define a *long λ-approximation sequence* to be a retrospective sequence $(M_\alpha)_{\alpha<\eta}$ of elementary substructures of $H(\theta)$ such that $|M_\alpha| < \lambda$ and $\lambda \cap M_\alpha \in \lambda$ for all α.

Requiring $\lambda \cap M_\alpha \in \lambda$ is equivalent to requiring that if $X \in M_\alpha$ and $|X| < \lambda$, then $X \subset M_\alpha$.

To prove the Fundamental Lemma for long λ-approximation sequences, simply replace ω_1 with λ in the proof of the lemma and in the definition of cardinal normal form.